Sec 6.2, 6.6 Two Dimensional Constant Linear Systems (Phase Plane)

In this section we will only consider linear systems of the form $\vec{Y}' = A\vec{Y}$, where A is an invertible 2×2 . These systems can be written as

$$\left[\begin{array}{c} x'\\y'\end{array}\right] = \left[\begin{array}{c} a & b\\c & d\end{array}\right] \left[\begin{array}{c} x\\y\end{array}\right]$$

$$\vec{y}'(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$
$$\vec{y}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

where $det(A) \neq 0$.

How to sketch the solutions of this system?

Ex1. Consider the following first order linear system: $\begin{bmatrix} x'\\y' \end{bmatrix} = \begin{bmatrix} -1 & 2\\ 0 & -3 \end{bmatrix} \begin{bmatrix} x\\y \end{bmatrix}$ The general solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \mathbf{c}_1 \mathbf{e}^{-1t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{c}_2 \mathbf{e}^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Notice that the eigenvalues are real and negative.

Step1: Sketch the four "easy" solutions; i.e. the solutions that correspond to the pairs:

\mathbf{c}_1	\mathbf{c}_2
1	0
-1	0
0	1
0	-1

For example, if $c_1 = 1$ and $c_2 = 0$, then we can identify the corresponding solution with the pair

$$(x(t), y(t)) = \mathbf{e}^{-t}(1, 0)$$

This is the parametric curve $x(t) = e^{-t}$, y(t) = 0. The **trace** of this parametric curve is the positive x-axis.

For example, if $c_1 = 0$ and $c_2 = 1$, then we can identify the corresponding solution with the pair

$$(x(t), y(t)) = \mathbf{e}^{-3t}(1, -1).$$

This is the parametric curve $x(t) = e^{-3t}$, $y(t) = -e^{-3t}$. The **trace** of this parametric curve is the open ray y = -x whose end point at the origin and contains the point (1, -1).

Step2: Fill in the rest. How?

We need to understand the behavior of the solutions. Since **both eigenvalues are negative**, regardless the values of \mathbf{c}_1 and \mathbf{c}_2 any solution converges to the origin point (0,0). But we can say more about the behavior of the solutions.

- As $t \to \infty$, the dominant term is $c_1 \mathbf{e}^{-1t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. In other words, for large t, any solution is approximately parallel to $\mathbf{e}^{-1t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- As $t \to -\infty$, the dominant term is $c_2 \mathbf{e}^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. In other words, for large t (negative), any solution is approximately parallel to $\mathbf{e}^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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If the eigenvalues are real and negative we say that the origin is a nodal sink (sink node), or asymptotically stable node.

Ex. 1 (page above)

$$\vec{y}' = A \vec{y}$$
 $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$ Find Figm pairs
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Ex2. Sketch the solutions of the system $\begin{bmatrix} x'\\y' \end{bmatrix} = \begin{bmatrix} 1 & -2\\0 & 3 \end{bmatrix} \begin{bmatrix} x\\y \end{bmatrix}$.

Sol: One sees that the general solution is given by

If the eigenvalues are real and positive we say that the origin is a nodal source (source node), or unstable node.

Ex.3 Sketch the solutions of the system $\begin{bmatrix} x'\\y' \end{bmatrix} = \begin{bmatrix} -1 & 3\\ 5 & -3 \end{bmatrix}$	$\left[\begin{array}{c} x\\ y \end{array} \right]$
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If the eigenvalues are real and have opposite signs we say that the origin is a saddle point.

Ex3. Eign values
$$P(\lambda) = \det \begin{pmatrix} -1-\lambda & 3\\ 5 & -3-\lambda \end{pmatrix} = 0$$

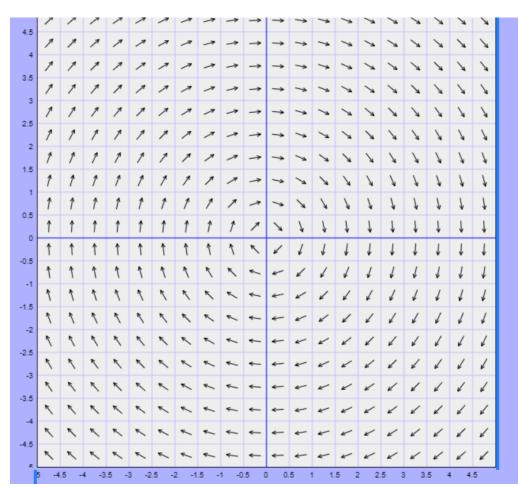
 $P(\lambda) = (1+\lambda)(3+\lambda) - 15 = 0 \iff P(\lambda) = (\lambda+6)(\lambda-2) \stackrel{n}{\Rightarrow} \lambda_{\lambda} = 2 \Rightarrow \lambda_{\lambda}$
 $\chi^{2} + 4\chi - 3\chi^{2}$
 $\chi^{2} + 4\chi - 12$
 $\hat{\gamma}(t) = c_{1}e^{-\delta t} \frac{1}{v_{1}} + c_{2}e^{-2\delta t} \frac{1}{v_{2}}$
 $\hat{\psi}_{1} = \begin{pmatrix} 1\\ -5\chi^{2} \end{pmatrix} \quad \hat{\psi}_{2} = \begin{pmatrix} 1\\ 1 \end{pmatrix}$

Ex.4 Sketch the solutions of the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

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Sol. The eigenvalues are -2 + 1i and -2 - 1i.

Ex.5 Sketch the solutions of the system $\begin{bmatrix} x'\\y' \end{bmatrix} = \begin{bmatrix} 0 & 3\\ -3 & 0 \end{bmatrix} \begin{bmatrix} x\\y \end{bmatrix}$.



Sol. The eigenvalues are 3i and -3i.

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Summary. Let A be an invertible 2×2 matrix. Classification of the origin for $\vec{Y}' = A\vec{Y}$.

- Real eigenvalues, both negative: asymptotically stable node, or sink node.
- Real eigenvalues, both positive: unstable node, or source node.
- Real eigenvalues of opposite sign: saddle point.
- Complex eigenvalues, $a \pm bi$ with a < 0: asymptotically stable focus.
- Complex eigenvalues, $a \pm bi$ with a > 0: unstable focus.
- Complex eigenvalues, $a \pm bi$ with a = 0: center.